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MONOTONE DIFFERENCE APPROXIMATIONS FOR SCALAR CONSERVATION LAWS--ETC(U)

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ABSTRACT

A complete self-contained treatment of the stability and convergence properties of conservation form, monotone difference approximations to scalar conservation laws in several space variables is developed. In particular, the authors prove that general monotone difference schemes always converge and that they converge to the physical weak solution satisfying the entropy condition. Rigorous convergence results follow for dimensional splitting algorithms when each step is approximated by a monotone difference scheme.

The results are general enough to include, for instance, Godunov's scheme, the upwind scheme (differenced through stagnation points), and the Lax-Friedrichs scheme together with appropriate multi-dimensional generalizations.

AMS (MOS) Subject Classifications: 65M10, 65M05.

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SIGNIFICANCE AND EXPLANATION

The class of partial differential equations called conservation laws arise in many areas of continuum mechanics including gas dynamics and fluid flows. Their solutions, even if initially smooth, develop jumps or "shock waves"; a phenomenon which occurs in many systems and is important in engineering applications.

It is desirable to have as complete an understanding as possible of the simplest case of a single scalar conservation law. This paper discusses the approximation of these equations by a broad class of difference schemes whose solutions are shown to converge to the desired solution of the equation.

MONOTONE DIFFERENCE APPROXIMATIONS FOR SCALAR CONSERVATION LAWS

Michael G. Crandall and Andrew Majda*

Introduction.

Perhaps the simplest mathematical models exhibiting behaviour typical of that encountered in inviscid continuum mechanics are the initial-value problems for a scalar conservation law. These problems are of the form

$$(0.1)\qquad \begin{cases} \text{(i)}\ u_t + \sum_{i=1}^{N} f_i(u)_{x_i} = 0, \quad \text{for} \quad t > 0,\ x = (x_1,\ldots,x_N) \in \mathbb{R}^N, \\ \text{(ii)}\ u(x,0) = u_0(x), \quad \text{for} \quad x \in \mathbb{R}^N, \end{cases}$$

where the f_i are smooth real-valued functions and u is a scalar. It is well known (see [14]) that even if the initial value u_0 is smooth, the solution to (0.1) typically develops discontinuities as t increases to some $t_0 > 0$ (i.e. shock waves form). Thus the differential equation must be understood in a generalized or weak sense. However, there can be an infinite number of generalized solutions of (0.1) with the same initial data u_0 and an additional principle, the entropy condition, is needed to select the unique "physical" weak solution (see [14]).

The main new result of this work establishes the convergence of general conservation form, monotone difference approximations to (0.1) to the unique generalized solution which satisfies the entropy condition. For notational simplicity in the sequel we restrict the presentation to the case $N = 2$ for the most part. The corresponding definitions and results for the general case will be clear from this. For $N = 2$ we write (x,y) rather than (x_1, x_2). Selecting mesh sizes $\Delta x,\ \Delta y,\ \Delta t > 0$, the value of our numerical approximation at $(j\Delta x, k\Delta y, n\Delta t)$ will be denoted by $U^n_{j,k}$. Capital letters U, V, etc. will denote functions on the x,y lattice $\Delta = \{(j\Delta x, k\Delta y) : j,k \text{ are integers}\}$ and the value of U at $(j\Delta x, k\Delta y)$ will be written $U_{j,k}$. Thus U^n, the state of our numerical approximation at the level $n\Delta t$, is a function on Δ with values $U^n_{j,k}$.

The standard notations $\lambda^x = \Delta t/\Delta x, \lambda^y = \Delta t/\Delta y$, $(\Delta^x_+ U)_{j,k} = U_{j+1,k} - U_{j,k}$, $(\Delta^y_+ U)_{j,k} = U_{j,k+1} - U_{j,k}$, etc., will be used. The difference approximations of (0.1) of interest here are explicit marching schemes of the form

$$(0.2)\qquad U^{n+1}_{j,k} = G(U^n_{j-p,k-s},\cdots,U^n_{j+q+1,k+r+1})$$

where p, s, q, r are nonnegative integers and G is a function of $(p+q+2)(r+s+2)$ real variables. (We are ignoring λ^x, λ^y dependence for the moment, as the quantities will typically be fixed.) To simplify notation, (0.2) will be written as

$$(0.3)\qquad U^{n+1} = \bar{G}(U^n),$$

with the choice of p, s, q, r dictated by the context. The difference approximation (0.2) - (0.3) said to have <u>conservation form</u> (see [10]) if there are functions g_1, g_2 such that

$$(0.4)\qquad \begin{aligned} \bar{G}(U)_{j,k} &= G(U_{j-p,k-s},\cdots,U_{j+q+1,k+s+1}) \\ &= U_{j,k} - \lambda^x\Delta^x_+ g_1(U_{j-p,k-s},\cdots,U_{j+q,k+s+1}) \\ &\quad - \lambda^y\Delta^y_+ g_2(U_{j-p,k-r},\cdots,U_{j+q+1,k+s}) . \end{aligned}$$

In order that (0.3) be <u>consistent</u> with (0.1) when (0.4) holds we must have

$$(0.5)\qquad g_i(u,\cdots,u) = f_i(u) \quad \text{for} \quad u \in \mathbb{R} \quad \text{and} \quad i = 1,2 .$$

The functions g_i are called the <u>numerical fluxes</u> of the approximation. Finally, the difference approximation is <u>monotone</u> on the interval $[a,b]$ if $G(a_1,\cdots,a_{(p+q+2)(r+s+2)})$ is a nondecreasing function of each argument a_i so long as all arguments lie in $[a,b]$.

It follows from the results of [13] that for $u_0 \in L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ there is a unique weak solution $u \in L^\infty(\mathbb{R}^2 \times [0,\infty))$ which satisfies the entropy condition of [13]. (See Section 2 for more details.) Moreover, we can write $u(x,y,t) = (S(t)u_0)(x,y)$ where $S(t) : L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \to L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ for each $t \geq 0$ and $t \to S(t)u_0$ is continuous into $L^1(\mathbb{R}^2)$. To compute this solution numerically we set

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$$U^0_{j,k} = \frac{1}{\Delta x \Delta y} \int_{R_{j,k}} u_0(x,y)\,dx\,dy \quad , \tag{0.6}$$

where $R_{j,k} = [(j - 1/2)\Delta x, (j + 1/2)\Delta x) \times [(k - 1/2)\Delta y, (k + 1/2)\Delta y)$, and define U^{n+1} from U^n by (0.3). Finally, put $\chi^n_{j,k}$ = characteristic function of $R_{j,k} \times [n\Delta t, (n+1)\Delta t)$ and

$$u^{\Delta t} = \sum_{n=0}^{\infty} \sum_{j,k=-\infty}^{\infty} U^n_{j,k}\, \chi^n_{j,k} \quad . \tag{0.7}$$

The main result is:

<u>Theorem 1</u>. Suppose $u_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $a \le u_0 \le b$ a.e. Let (0.2) be a consistent conservation form difference approximation to (0.1)(i) which is monotone on $[a,b]$ and which has Lipschitz continuous numerical fluxes g_i, $i = 1,2$. Let $u^{\Delta t}$ be given by (0.2), (0.6), (0.7). Then as $\Delta t \to 0$ with λ^x, λ^y fixed, $u^{\Delta t}$ converges to $S(t)u_0$ in $L^1(\mathbb{R}^2)$ uniformly for bounded $t \ge 0$. More precisely

$$\lim_{\Delta t \to 0} \sup_{0 \le t \le T} \iint_{\mathbb{R}^2} |u^{\Delta t}(x,y,t) - S(t)u_0(x,y)|\,dx\,dy = 0 \tag{0.8}$$

for each $T > 0$.

Reviewing the definitions, (0.8) can also be restated as

$$\lim_{\Delta t \to 0} \sup_{\substack{0 \le t \le T \\ n\Delta t \le t < (n+1)\Delta t}} \sum_{j,k} \int_{R_{j,k}} |U^n_{j,k} - S(t)u_0(x,y)|\,dx\,dy = 0 \quad . \tag{0.9}$$

It should be recalled that even if $N = 1$ and f_1 is convex, non-monotone schemes such as the Lax-Wendroff scheme, can converge to solutions which violate the entropy condition (see [10] and [18]). The result of Theorem 1 applies to the popular dimensional splitting algorithms (see [8], [17], [21]). This follows from simple observations. For example, consider the one-dimensional conservation laws

$$\begin{cases} \text{(i)} \quad v_t + f_1(v)_x = 0 \\ \text{(ii)} \quad w_t + f_2(w)_y = 0 \quad . \end{cases} \tag{0.10}$$

If

$$\begin{cases} \text{(i)} \quad V^{n+1}_j = G_1(V^n_{j-p}, \ldots, V^n_{j+q+1}) \\ \text{(ii)} \quad W^{n+1}_k = G_2(W^n_{k-r}, \ldots, W^n_{k+s+1}) \end{cases} \tag{0.11}$$

are conservation form difference approximations of (0.10)(i),(ii) respectively, Lax has observed that the scheme defined by

$$\begin{cases} U^{n+1}_{j,k} = G_1(U^{n+1/2}_{j-p,k}, \ldots, U^{n+1/2}_{j+q+1,k}) \\ U^{n+1/2}_{\ell,m} = G_2(U^n_{\ell,m-r}, \ldots, U^n_{\ell,m+s+1}) \end{cases} \tag{0.12}$$

has conservation form and is consistent with (0.1) (see [2] or check the definitions). When G_1 and G_2 are also monotone on $[a,b]$ the scheme in (0.12) will be monotone on $[a,b]$. (Since $a \le V_{j+\ell} \le b$ for $-p \le \ell \le q+1$ implies $a \le G_1(V_{j-p}, \ldots, V_{j+q+1}) \le b$ by Proposition 3.1(b) below, this is evident.) Thus Theorem 1 applies to the split scheme (0.12). This fact, along with other results concerning splitting algorithms is developed in [17].

The plan of this work is as follows: In Section 1 various monotone difference schemes to which Theorem 1 applies are recalled. These include the Lax-Friedrichs scheme, the upwind scheme (differenced through stagnation points) and Godunov's scheme. The construction of a wide variety of multi-dimensional schemes from the above one dimensional ones is discussed. In view of these examples, most of the results of LeRoux [16] for specific schemes in a single space dimension are included in our general approach. Section 2 is a review of some basic facts about solutions to (0.1) and some function spaces and estimates needed in the proof of Theorem 1. We discuss the stability of monotone conservation form difference schemes in Section 3. In particular, we prove that such schemes define L^1 contractions; this was proved by Jennings [11] in the case of one space variable but even there our proof, based on a lemma of Crandall and Tartar [5] is simpler. In Section 4 we verify that if solutions of monotone conservation form difference approximations converge, the limit satisfies the entropy conditions. This was proved by Harten, Hyman and Lax in [10] for $N = 1$; however we build a different discrete entropy flux which yields a simpler proof for general N and requires only

continuity of the numerical fluxes. This generality is useful for applications. (See Section 1 and [16].) The various results of Sections 2, 3, 4 are pieced together to prove Theorem 1 in Section 5. In Section 6 we briefly discuss the inhomogeneous equation.

In fact our arguments yield more than Theorem 1 states, for the existence of the solution $S(t)u_0$ of (0.1) is established while proving convergence (see Section 5). See Conway and Smoller [3], Douglis [6], and Kojima [12] for earlier used of the (monotone) Lax-Friedrichs difference approximation to prove existence. Oharu and Takahashi prove this scheme converges to the solution satisfying the entropy condition via nonlinear semigroup methods in [19].

Section 1. Examples

In this section we present a variety of difference schemes to which Theorem 1 applies. Later sections are independent of the current one.

We begin with several well-known schemes in the case $N = 1$. So long as $N = 1$ we will write f, g in place of f_1, g_1. For a single-space variable the Lax-Friedrichs scheme is given by

(1.1) $$u_j^{n+1} = u_j^n - \frac{\lambda^x}{2}\Delta_0^x f(u_j^n) + \frac{1}{2}\Delta_+^x \Delta_-^x u_j^n \ .$$

Equivalently,

$$u_j^{n+1} = u_j^n - \lambda^x \Delta_+^x g(u_j, u_{j-1})$$

where the numerical flux g is given by

$$g(u_j, u_{j-1}) = \frac{f(u_j) + f(u_{j-1})}{2} - \frac{1}{2\lambda^x}(u_j - u_{j-1}) \ .$$

The conservation form and consistency are obvious. A simple analysis reveals that (1.1) is also monotone on $[a,b]$ provided that the C.F.L. condition

(1.2) $$\lambda^x \max_{a\le u\le b} |f'(u)| \le 1$$

holds. More generally, if γ is nondecreasing, the scheme

(1.3) $$u_j^{n+1} = u_j^n - \frac{\lambda^x}{2}\Delta_0^x f(u_j^n) + \frac{1}{2}\Delta_+^x \Delta_-^x \gamma(u_j^n)$$

has conservation form with the flux

$$g(u_j, u_{j-1}) = \frac{f(u_j) + f(u_{j-1})}{2} - \frac{1}{2\lambda^x}(\gamma(u_j) - \gamma(u_{j-1}))$$

and (1.3) is monotone on $[a,b]$ provided that

(1.4) $$1 - \lambda^x \gamma'(u) \ge 0 \quad \text{and} \quad \gamma'(u) - \lambda^x |f'(u)| \ge 0 \quad \text{for} \quad a \le u \le b \ .$$

If f is nondecreasing, the upwind difference scheme is given by

(1.5) $$u_j^{n+1} = u_j^n - \lambda^x (f(u_j^n) - f(u_{j-1}^n)) = u_j^n - \lambda^x \Delta_+^x f(u_{j-1}^n)$$

while if f is nonincreasing it has the form

$$u_j^{n+1} = u_j^n - \lambda^x \Delta_+^x f(u_j^n) \quad . \tag{1.6}$$

Next let $\alpha \in \mathbb{R}$ and assume

$$f'(u) \geq 0 \quad \text{for} \quad u \geq \alpha \quad \text{and} \quad f'(u) \leq 0 \quad \text{for} \quad u \leq \alpha \quad . \tag{1.7}$$

Thus $f'(\alpha) = 0$ and α is a <u>stagnation point</u>. Without loss of generality, we may also assume that $f(\alpha) = 0$ (since changing f by a constant leaves (0.1) invariant). Set

$$\theta(s) = \begin{cases} 1 & \text{if} \quad s \leq \alpha \\ 0 & \text{if} \quad s > \alpha \end{cases} \tag{1.8}$$

and consider the scheme

$$u_j^{n+1} = u_j^n - \lambda^x \Delta_+^x (\theta(u_j^n) f(u_j^n) + (1 - \theta(u_{j-1}^n)) f(u_{j-1}^n)) \quad . \tag{1.9}$$

The resulting scheme (for any choice of θ) is clearly consistent, has conservation form, and when $\theta \equiv 0$ or $\theta \equiv 1$ reduces to the conventional upwind schemes (1.5), (1.6). In our case (1.7) holds, θ given by (1.8) and $f(\alpha) = 0$), it is easy to verify that (1.10) is monotone on $[a,b]$ provided that (1.2) holds.

The last scheme we mention for $N = 1$ is <u>Godunov's method</u>. This method will be discussed in full together with certain generalizations in [17]. Here we consider only the case in which $f'' > 0$ (in particular, f is strictly convex). Godunov's method is then given by the three-point conservation form scheme

$$u_j^{n+1} = u_j^n - \lambda^x \Delta_+^x g(u_{j-1}^n, u_j^n) \tag{1.10}$$

where the numerical flux g is defined by the complicated recipe:

$$g(u_{j-1}, u_j) = \begin{cases} f(u_{j-1}) & \text{if} \quad u_{j-1} \geq u_j \quad \text{and} \quad f(u_{j-1}) \geq f(u_j) \\ f(u_j) & \text{if} \quad u_{j-1} \geq u_j \quad \text{and} \quad f(u_{j-1}) \leq f(u_j) \\ f(u_{j-1}) & \text{if} \quad u_{j-1} \leq u_j \quad \text{and} \quad f'(u_{j-1}) \geq 0 \\ f(u_j) & \text{if} \quad u_{j-1} \leq u_j \quad \text{and} \quad f'(u_j) \leq 0 \\ f((f')^{-1}(0)) & \text{otherwise} \ . \end{cases} \tag{1.11}$$

We have written down the function g explicitly because these formulas show that for Godunov's method the numerical flux is Lipschitz continuous but not everywhere differentiable. In order to verify that Godunov's method defines a monotone scheme we recall the basic idea of the method. Given bounded discrete data u_j^n with $a \leq u_j^n \leq b$, let

$$u_0 = \sum_{j=-\infty} u_j^n \chi_j$$

where χ_j is the characteristic function of $(j - 1/2)\Delta x \leq x < (j + 1/2)\Delta x$. Now we let u be the exact solution of

$$\begin{cases} u_t + f(u)_x = 0 \\ u(0,x) = u_0(x) \quad , \end{cases} \tag{1.12}$$

which we write as $u(x,t) = S(t)u_0(x)$. Since solutions of (1.11) propogate at finite speed at most $c_0 = \max\limits_{a \leq u \leq b} |f'(u)|$ (see Section 2), it follows that $S^x(\Delta t)u_0(x)$, restricted to $(j - 1/2)\Delta x \leq x \leq (j + 1/2)\Delta x$, depends only on the three values $u_{j-1}^n, u_j^n, u_{j+1}^n$ provided that, as we now assume, (1.2) holds. Using the form of the exact solution of (1.12) for $0 \leq t \leq \Delta t$ (see [14]) and the standard flux balance relation, one explicitly computes

$$u_j^{n+1} = \frac{1}{\Delta x} \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} S^x(\Delta t) u_0(x) dx$$

and finds (1.10), (1.11). It is well known (and see our proof of Theorem 1) that $v_0 \geq u_0$ implies $S(t)v_0 \geq S(t)u_0$. Hence u_j^{n+1} is a nondecreasing function of $u_{j-1}^n, u_j^n, u_{j+1}^n$, which establishes the monotonicity of the scheme.

Multi-dimensional examples of schemes to which Theorem 1 applies can be built from the one dimensional schemes in a variety of ways. First there are the methods of dimensional splitting as mentioned in the introduction. To this we add one additional method. Let

$$\begin{aligned} &(i) \quad v_j^{n+1} = G_1(v_{j-p}^n, \ldots, v_{j+q+1}^n) = v_j^n - \lambda^x \Delta_+^x g_1(v_{j-p}^n, \ldots, v_{j+q}^n) \\ &(ii) \quad w_k^{n+1} = G_2(w_{k-r}^n, \ldots, w_{k+s+1}^n) = w_j^n - \lambda^y \Delta_+^y g_2(w_{k-r}^n, \ldots, w_{k+s+1}^n) \end{aligned} \tag{1.13}$$

denote two conservation form schemes consistent with

$$(1.14)\qquad \begin{cases} \text{(i)} \quad v_t + (\frac{1}{\alpha} f_1(v))_x = 0 \\ \text{(ii)} \quad w_t + (\frac{1}{1-\alpha} f_2(w))_y = 0 \end{cases}$$

respectively, where $0 < \alpha < 1$. Now form the composite scheme

$$(1.15)\qquad U^{n+1}_{j,k} = \alpha\, G_1(U^n_{j-p,k},\dots,U^n_{j+q+1,k}) + (1-\alpha)G_2(U^n_{j,k-r},\dots,U^n_{j,k+s+1}) \ .$$

Then (1.15) has conservation form and is consistent with $u_t + f_1(u)_x + f_2(u)_y = 0$. Moreover, if (1.13)(i), (ii) are monotone on $[a,b]$ so then is (1.15). Any of the schemes discussed above may be used for G_1, G_2. For example, if $\alpha = 1/2$ and G_1, G_2 are chosen as Lax-Friedrich's scheme (1.1), then (1.15) reads

$$u^{n+1}_{j,k} = u^n_{j,k} - \frac{\lambda^x}{2}\Delta^x_0 f_1(u^n_{j,k}) - \frac{\lambda^y}{2}\Delta^y_0 f_2(u^n_{j,k}) + \frac{1}{4}(\Delta^x_+\Delta^x_- + \Delta^y_+\Delta^y_-)(u^n_{j,k})$$

which is the Lax-Friedrich's scheme in two dimensions. The C.F.L. condition guaranteeing monotonicity on $[a,b]$ is now

$$\lambda^x \max_{a\le u\le b} |f_1'(u)|,\ \lambda^y \max_{a\le u\le b} |f_2'(u)| \le \frac{1}{2} \ .$$

which is a factor of 2 more severe than (1.2).

Section 2. Preliminaries

We begin by recalling some of the basic facts concerning solutions of the problem (0.1). A <u>weak solution</u> of the conservation law (0.1)(i) on $\mathbb{R}^N \times [0,T]$ is a function $u \in L^\infty(\mathbb{R}^N \times [0,T])$ such that

$$(2.1)\qquad \begin{cases} \int_0^T \int_{\mathbb{R}^N} (\varphi_t u + \sum_{i=1}^{N} \varphi_{x_i} f_i(u))\,dx\,dt = 0 \\ \text{for every} \quad \varphi \in C^1_0(\mathbb{R}^N \times (0,T)) \end{cases}$$

where $C^1_0(\mathbb{R}^N \times (0,T))$ denotes the continuously differentiable functions on $\mathbb{R}^N \times (0,T)$ with compact support. As we remarked earlier, weak solutions are not uniquely determined by their initial data (properly interpreted), and an additional condition, the entropy condition, is needed to select the desired solution. The form of the entropy condition we will use was given by Vol'pert [22]. An <u>entropy solution</u> of the conservation law (0.1)(i) on $\mathbb{R}^N \times (0,T)$ is a function $u \in L^\infty(\mathbb{R}^N \times [0,T])$ such that

$$(2.2)\qquad \begin{cases} \int_0^T \int_{\mathbb{R}^N} \varphi_t |u-c| + \sum_{i=1}^{N} \varphi_{x_i} \operatorname{sgn}(u-c)(f_i(u) - f_i(c))\,dx\,dt \ge 0 \\ \text{for every} \quad \varphi \in C^1_0(\mathbb{R}^2 \times (0,T)) \quad \text{with} \quad \varphi \ge 0 \quad \text{and every} \quad c \in \mathbb{R} \ . \end{cases}$$

In (2.2), $\operatorname{sgn} r = r/|r|$ for $r \neq 0$ (and the value assigned to $\operatorname{sgn} 0$ is irrelevant since $f_i(u) - f_i(c) = 0$ if $u = c$). If $a \le u \le b$ a.e., then choosing $c = b$ and $c = a$ in (2.2) we can deduce (2.1) for $\varphi \ge 0$ (and hence in general). That is, entropy solutions are weak solutions. Moreover, if $a \le u \le b$ a.e. and (2.2) holds for $a \le c \le b$, then it clearly holds for all c. The existence and uniqueness of entropy solutions of (0.1) which are of (locally) bounded variation (see below) when u_0 is of (locally) bounded variation and the initial condition is properly interpreted was proved by Vol'pert [22]. Subsequently Kružkov [13] extended these results. A special case of Kružkov's uniqueness theorem adequate for our purposes is:

<u>Theorem 2.1</u>. (Uniqueness) Let $u_{10}, u_{20} \in L^\infty(\mathbb{R}^N)$. Let u_1, u_2 be entropy solutions of (0.1)(i) on $\mathbb{R}^N \times (0,T)$ for which

$$(2.3)\qquad \operatorname*{essential\ limit}_{t\downarrow 0} \int_{|x|\le R} |u_i(x,t) - u_{i0}(x)|\,dx = 0 \quad \text{for} \quad R > 0,\ i = 1,2 \ .$$

Let L be a Lipschitz constant for the mapping $r \to (f_1(r),\cdots,f_N(r))$ on $|r| \ge \max(\|u_i\|_{L^\infty(\mathbb{R}^N\times(0,T))} : i = 1,2)$. Then

$$(2.4)\qquad \begin{cases} \int_{|x|\le R} |u_1(x,t) - u_2(x,t)|\,dx \le \int_{|x|\le R+Lt} |u_{10}(x) - u_{20}(x)|\,dx \\ \text{for } R > 0 \text{ and almost all } t \in [0,T] \ . \end{cases}$$

In particular, if $u_{10} = u_{20}$, then $u_1 = u_2$ a.e. If we choose $u_{20} = u_2 = 0$ in (2.4) (constants are entropy solutions of (0.1)(i)) we find

$$\int_{|x|\le R} |u_1(x,t)|\,dx \le \int_{|x|\le R+Lt} |u_{10}(x)|\,dx \ .$$

If $u_{10} \in L^1(\mathbb{R}^N)$, letting $R \to \infty$ above yields

$$\int_{\mathbb{R}^N} |u_1(x,t)|\,dx \le \int_{\mathbb{R}^N} |u_{10}(x)|\,dx \ ,$$

so $t \to u_1(\cdot,t)$ is bounded into $L^1(\mathbb{R}^N)$ (or $u_1 \in L^\infty(0,T : L^1(\mathbb{R}^N))$). It will prove convenient for us to work with this case, that is $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. It is a simple matter to pass then to the more general case $u_0 \in L^\infty(\mathbb{R}^N)$ via the finite domain of dependence established in (2.4) (or the finite numerical domain of dependence for our schemes).

It follows from the results of [13], [22] that for every $u_0 \in L^\infty(\mathbb{R}^N)$ (0.1)(i) has an entropy solution u assuming the initial-value u_0 in the sense (2.3). This existence result will follow easily from our inventigations, so we will not belabor it here.

Next we define some function spaces needed in the sequel. From now on we set $N = 2$ in this section and assume the reader can extrapolate to the case of general N. $L^1_{loc}(\mathbb{R}^2)$ is the space of functions which are integrable on compact subsets of $\mathbb{R}^2$. A sequence $\{g_n\} \subseteq L^1_{loc}(\mathbb{R}^2)$ converges to g in $L^1_{loc}(\mathbb{R}^2)$ provided $\|g_n - g\|_{L^1(K)} \to 0$ for every compact $K \subset \mathbb{R}^2$. $BV(\mathbb{R}^2)$ denotes the subspace of $f \in L^1_{loc}(\mathbb{R}^2)$ for which $\|f\|_{BV(\mathbb{R}^2)} < \infty$ where

$$(2.5)\qquad \|f\|_{BV(\mathbb{R}^2)} = \sup_{h\in\mathbb{R}\setminus\{0\}} \int_{\mathbb{R}^2} \frac{|f(x+h,y) - f(x,y)|}{|h|}\,dxdy + \sup_{h\in\mathbb{R}\setminus\{0\}} \int_{\mathbb{R}^2} \frac{|f(x,y+h) - f(x,y)|}{|h|}\,dxdy \ .$$

We observe that for $f \in BV(\mathbb{R}^2)$ and $h, s \in \mathbb{R}$

$$(2.6)\qquad \begin{aligned} &\int_{\mathbb{R}^2} |f(x+h,y+s) - f(x,y)|\,dxdy \\ &\le \int_{\mathbb{R}^2} |f(x+h,y+s) - f(x,y+s)|\,dxdy + \int_{\mathbb{R}^2} |f(x,y+s) - f(x,y)|\,dxdy \\ &\le (|h| + |s|)\|f\|_{BV(\mathbb{R}^2)} \ . \end{aligned}$$

The space $L^1(\mathbb{R}^2) \cap BV(\mathbb{R}^2)$ is equipped with the norm

$$(2.7)\qquad \|f\|_{L^1(\mathbb{R}^2)\cap BV(\mathbb{R}^2)} = \|f\|_{L^1(\mathbb{R}^2)} + \|f\|_{BV(\mathbb{R}^2)} \ .$$

We will use the following compactness criterion (which is stated with unnecessarily restrictive hypotheses).

<u>Proposition 2.2</u>. Let $\mathcal{K} \subset L^1(\mathbb{R}^2)$ be bounded and

$$(2.8)\qquad \lim_{(h,s)\to(0,0)} \sup_{f\in\mathcal{K}} \int_{\mathbb{R}^2} |f(x+h,y+s) - f(x,y)|\,dxdy = 0 \ .$$

Then $\mathcal{K}$ is precompact in $L^1_{loc}(\mathbb{R}^2)$.

This is immediate from standard results, e.g., [6,IV.8.21]. As a corollary we note that (2.6), (2.7) imply

(2.9) bounded subsets of $L^1(\mathbb{R}^2) \cap BV(\mathbb{R}^2)$ are precompact in $L^1_{loc}(\mathbb{R}^2)$.

Finally we summarize several facts which will be used in passing between continuous and discrete estimates. Let us recall the notations. U, V, etc. denote functions on the lattice $\Delta = \{(j\Delta x, k\Delta y) = \Delta t(j/\lambda^x, b/\lambda^y) : j,k \text{ integers}\}$ with values $U_{j,k}$, $V_{j,k}$, etc. We set

$$(2.10)\qquad \|U\|_{L^1(\Delta)} = \sum_{j,k} |U_{j,k}|\Delta x\Delta y, \quad \|U\|_{L^\infty(\Delta)} = \sup_{j,k}|U_{j,k}|$$

and

$$(2.11)\qquad \|U\|_{BV(\Delta)} = \frac{1}{\Delta x}\|\Delta^x_+ U\|_{L^1(\Delta)} + \frac{1}{\Delta y}\|\Delta^y_+ U\|_{L^1(\Delta)}$$

where $(\Delta^x_+ U)_{j,k} = U_{j+1,k} - U_{j,k}$, etc. $L^1(\Delta)$, $BV(\Delta)$ consist of those U for which (2.10),

(2.11) are finite.

The characteristic function of the rectangle $R_{j,k} = [(j-1/2)\Delta x, (j+1/2)\Delta x) \times [(k-1/2)\Delta y, (k+1/2)\Delta y)$ is denoted by $\chi_{j,k}$. Given a lattice function U we define a piecewise constant function $U_{\mathbb{R}^2}$ on $\mathbb{R}^2$ by $U_{\mathbb{R}^2} = U_{j,k}$ on $R_{j,k}$. Equivalently,

$$U_{\mathbb{R}^2} = \sum_{j,k} U_{j,k}\chi_{R_{j,k}} . \tag{2.12}$$

Given a function $u \in L^1_{loc}(\mathbb{R}^2)$ we define a lattice function u_Δ by

$$(u_\Delta)_{j,k} = \frac{1}{\Delta x \Delta y}\int_{R_{j,k}} u(x,y)\,dxdy . \tag{2.13}$$

One has:

$$\|U_{\mathbb{R}^2}\|_{L^1(\mathbb{R}^2)} = \|U\|_{L^1(\Delta)},\ \|U_{\mathbb{R}^2}\|_{L^\infty(\mathbb{R}^2)} = \|U\|_{L^\infty(\Delta)} \tag{2.14}$$

$$\|u_\Delta\|_{L^1(\Delta)} \le \|u\|_{L^1(\mathbb{R}^2)},\ \|u_\Delta\|_{L^\infty(\Delta)} \le \|u\|_{L^\infty(\mathbb{R}^2)} , \tag{2.15}$$

$$\|U_{\mathbb{R}^2}\|_{BV(\mathbb{R}^2)} \le \|U\|_{BV(\Delta)} , \tag{2.16}$$

and

$$\|u_\Delta\|_{BV(\Delta)} \le \|u\|_{BV(\mathbb{R}^2)} . \tag{2.17}$$

of these relations, (2.14) and (2.15) are immediate from the definitions. Also, (2.17) is easy to see for, by the definitions,

$$\begin{aligned}\|u_\Delta\|_{BV(\Delta)} &= \sum_{j,k}\Big(\frac{1}{\Delta x}\Big|\int_{R_{j+1,k}} u(x,y)\,dxdy - \int_{R_{j,k}} u(x,y)\,dxdy\Big| \\ &\quad + \frac{1}{\Delta y}\Big|\int_{R_{j,k+1}} u(x,y)\,dxdy - \int_{R_{j,k}} u(x,y)\,dxdy\Big|\Big) \\ &= \sum_{j,k}\Big(\frac{1}{\Delta x}\Big|\int_{R_{j,k}} (u(x+\Delta x, y) - u(x,y))\,dxdy\Big| + \frac{1}{\Delta y}\Big|\int_{R_{j,k}} u(x, y+\Delta y) - u(x,y)\,dxdy\Big|\Big) \\ &\le \|u\|_{BV(\mathbb{R}^2)} .\end{aligned}$$

It remains to check (2.17). To this end, observe that

$$\begin{aligned}\int_{\mathbb{R}^2}|f(x+h_1+h_2,y) - f(x,y)|\,dxdy &\le \int_{\mathbb{R}^2}|f(x+h_1+h_2,y) - f(x+h_2,y)|\,dxdy \\ &\quad + \int_{\mathbb{R}^2}|f(x+h_2,y) - f(x,y)|\,dxdy = \int_{\mathbb{R}^2}|f(x+h_1,y) - f(x,y)|\,dxdy \\ &\quad + \int_{\mathbb{R}^2}|f(x+h_2,y) - f(x,y)|\,dxdy .\end{aligned} \tag{2.18}$$

That is $g(h_1+h_2) \le g(h_1) + g(h_2)$ where $g(h_1+h_2)$ is the left-hand side of (2.18). Assume now that $h > 0$ and write $h = (\ell\Delta x + \xi\Delta x)$ where ℓ is a nonnegative integer and $0 \le \xi < 1$. By the above

$$\begin{aligned}\int_{\mathbb{R}^2}|U_{\mathbb{R}^2}(x+h, y) - U_{\mathbb{R}^2}(x,y)|\,dxdy &\le \ell\int_{\mathbb{R}^2}|U_{\mathbb{R}^2}(x+\Delta x, y) - U_{\mathbb{R}^2}(x,y)|\,dxdy \\ &\quad + \int_{\mathbb{R}^2}|U_{\mathbb{R}^2}(x+\xi\Delta x, y) - U_{\mathbb{R}^2}(x,y)|\,dxdy .\end{aligned} \tag{2.19}$$

By the definitions

$$\begin{aligned}\int_{\mathbb{R}^2}|U_{\mathbb{R}^2}(x+\Delta x, y) - U_{\mathbb{R}^2}(x,y)|\,dxdy &= \sum_{j,k}|U_{j+1,k} - U_{j,k}|\Delta x\Delta y \\ &= \|\Delta_+^x U\|_{L^1(\Delta)} .\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^2}|U_{\mathbb{R}^2}(x+\xi\Delta x, y) - U_{\mathbb{R}^2}(x,y)|\,dxdy \\ &= \sum_{j,k}\int_{R_{j,k}}|U_{\mathbb{R}^2}(x+\xi\Delta x, y) - U_{j,k}|\,dxdy \\ &= \sum_{j,k}\int_{(j+1/2-\xi)\Delta x}^{(j+1/2)\Delta x}\int_{(k-1/2)\Delta y}^{(k+1/2)(\Delta y)}|U_{j+1,k} - U_{j,k}|\,dydx \\ &= \sum_{j,k}\xi\,|U_{j+1,k} - U_{j,k}|\Delta x\Delta y .\end{aligned} \tag{2.21}$$

Combining (2.19), (2.20), (2.21) and recalling $h = \ell\Delta x + \xi\Delta x$ we find

$$\frac{1}{h}\int_{\mathbb{R}^2}|U_{\mathbb{R}^2}(x+h,y) - U_{\mathbb{R}^2}(x,y)|\,dxdy \le \frac{1}{\Delta x}\|\Delta_+^x U\|_{L^1(\Delta)} .$$

Treating the y-variation in a similar way gives (2.16).

Section 3. Stability Properties of Monotone Conservation Form Schemes

Recall that our schemes have the form

$$U^{n+1}_{j,k} = G(U^n_{j-p,k-r},\dots,U^n_{j+q+1,k+s+1}) \tag{3.1}$$

which we write as $U^{n+1} = \hat{G}(U^n)$. The first result is:

Proposition 3.1. Let (3.1) be a conservation form difference scheme which is monotone on the interval $[a,b]$. Let U, V satisfy $a \le U_{j,k}, V_{j,k} \le b$ for all j,k. Then:

(a) $\quad U \le V$ (i.e. $U_{j,k} \le V_{j,k}$ for all j,k) implies $\hat{G}(U) \le \hat{G}(V)$.

(b) $$\min_{\substack{j-p\le \ell\le j+q+1\\ k-r\le m\le k+s+1}} U_{\ell,m} \le \hat{G}(U)_{j,k} \le \max_{\substack{j-p\le \ell\le j+q+1\\ k-r\le m\le k+s+1}} U_{\ell,m} \quad \text{for all } j,k$$

(c) $$\|\hat{G}(U) - \hat{G}(V)\|_{L^1(\Delta)} \le \|U - V\|_{L^1(\Delta)} \quad \text{if } U, V \in L^1(\Delta)$$

(d) $$\|\hat{G}(U)\|_{BV(\Delta)} \le \|U\|_{BV(\Delta)} \quad \text{if } U \in L^1(\Delta) .$$

Proof. We will show first that (c) $\Rightarrow$ (d). Indeed, if $(\tau_{\Delta x}W)_{j,k} = W_{j+1,k}$ we have $\tau_{\Delta x}\hat{G}(U) = \hat{G}(\tau_{\Delta x}U)$ (that is $\hat{G}$ computes with grid translations). Then, by (c),

$$\begin{aligned}\|\Delta_+^x\hat{G}(U)\|_{L^1(\Delta)} &= \|\tau_{\Delta x}\hat{G}(U) - \hat{G}(U)\|_{L^1(\Delta)}\\ &= \|\hat{G}(\tau_{\Delta x}U) - \hat{G}(U)\|_{L^1(\Delta)}\\ &\le \|\tau_{\Delta x}U - U\|_{L^1(\Delta)} = \|\Delta_+^x U\|_{L^1(\Delta)} .\end{aligned}$$

Similarly, $\|\Delta_+^y\hat{G}(U)\|_{L^1(\Delta)} \le \|\Delta_+^y U\|_{L^1(\Delta)}$. Now (d) follows from the definition (2.11). A moment's reflection shows that (a) is merely a restatement of the assumption that (3.1) is monotone on $[a,b]$. We next verify (b). Letting $V_{j,k} = \sup_{\ell,m} U_{\ell,m} = c$ for all j,k we have $\hat{G}(V) = V$ ($\hat{G}$ maps constants to themselves by the conservation form). Now $V \ge U$ so $c = \hat{G}(V)_{j,k} \ge \hat{G}(U)_{j,k}$ for all j,k. Estimating below in a similar way yields $\inf_{\ell,m} U_{\ell,m} \le U_{j,k} \le \sup_{\ell,m} U_{\ell,m}$. The refinement (b) of this result follows from the fact that $G(U)_{j,k}$ depends only on $U_{\ell,m}$ for ℓ,m in the range indicated in (b).

It remains to prove (c). The following elementary lemma of M. Crandall and L. Tartar is at the heart of the matter. We use the notations $f\vee g = \max(f,g)$ and $f^+ = f\vee 0$.

Lemma 3.2. Let Ω be a measure space and $C \subset L^1(\Omega)$ have the property that $f\vee g \in C$ whenever $f, g \in C$. Let $T : C \to L^1(\Omega)$ satisfy

$$\int_\Omega T(f) = \int_\Omega f \quad \text{for } f \in C . \tag{3.2}$$

Then the following three properties of T are equivalent:

(a) $\quad f, g \in C$ and $f \le g$ a.e. implies $T(f) \le T(g)$ a.e.

(b) $$\int_\Omega (T(f) - T(g))^+ \le \int_\Omega (f - g)^+ \quad \text{for } f, g \in C$$

(c) $$\int_\Omega |T(f) - T(g)| \le \int_\Omega |f - g| \quad \text{for } f, g \in C .$$

The proof of Lemma 3.2 is given at the end of this section. We apply it here with $\Omega = \Delta$ equipped with the discrete measure assigning mass $\Delta x\Delta y$ to each point. Set

$$C = \{U \in L^1(\Delta) : a \le U_{j,k} \le b \text{ for all } j,k\} ,$$

and $T(U) = \hat{G}(U)$. It follows from Proposition 3.1(b) that

$$\|\hat{G}(U)\|_{L^1(\Delta)} \le (p+q+2)(s+r+2)\|U\|_{L^1(\Delta)} ,$$

so $\hat{G} : C \to L^1(\Delta)$. For $U \in C$ we have

$$\int_\Delta \hat{G}(U) = \sum_{j,k} G(U)_{j,k}\Delta x\Delta y = \sum_{j,k} U_{j,k}\Delta x\Delta y = \int_\Delta U$$

because of the conservation form of G. Thus (3.2) is verified in this case. Moreover, Lemma 3.2(a) is the same as Proposition 3.1(a) in this application. Thus Proposition 3.1(c) holds by the equivalence of Lemma 3.2(a) and (c).

Remark 3.3. In fact Proposition 3.1(c) and (d) hold even if $U, V \notin L^1(\Delta)$, as can easily be shown.

Next we examine the continuity properties in time of the function constructed as follows: Let $U^0 \in L^1(\Delta)$ be given with $a \le U^0_{j,k} \le b$. Set

$$U^{n+1} = \vec{G}(U^n) = \vec{G}^n(U^0) \quad \text{and}$$

$$u^{\Delta t} = \sum_{n=0}^{\infty} U^n_{\mathbb{R}^2}\chi^n = \sum_{n=0}^{\infty} \sum_{j,k} U^n_{j,k}\chi^n_{j,k} \tag{3.3}$$

where χ^n is the characteristic function of $[n\Delta t,(n+1)\Delta t)$ and $\chi^n_{j,k}$ is the characteristic function of $R_{j,k} \times [n\Delta t,(n+1)\Delta t)$.

<u>Proposition 3.4</u>. Let the assumptions of Proposition 3.1 hold. Let $u^{\Delta t}$ be given by (3.3) and $0 \le t_1 \le t_2$. Then

$$\int_{\mathbb{R}^2} |u^{\Delta t}(x,y,t_2) - u^{\Delta t}(x,y,t_1)|\,dxdy \le \left(\frac{|t_2 - t_1|}{\Delta t} + 1\right)\|\vec{G}(U^0) - U^0\|_{L^1(\Delta)} .$$

<u>Proof</u>. Let $n\Delta t \le t_2 < (n+1)\Delta t$, $m\Delta t \le t_1 < (m+1)\Delta t$ so that $u(\cdot,\cdot,t_2) = U^n_{\mathbb{R}^2}$, $u(\cdot,\cdot,t_1) = U^m_{\mathbb{R}^2}$.

We will show that

$$\|U^n_{\mathbb{R}^2} - U^m_{\mathbb{R}^2}\|_{L^1(\mathbb{R}^2)} \le (n-m)\|\vec{G}(U^0) - U^0\|_{L^1(\Delta)} .$$

from which (3.12) follows since $(n-m)\Delta t \le t_2 - t_1 + \Delta t$ by the choice of n and m. Using Proposition 3.1(b), $a \le U^n_{j,k} \le b$ for $n = 0,1,\cdots$ and all j,k. Then (2.14) and Proposition 3.1(c) yield

$$\|U^n_{\mathbb{R}^2} - U^m_{\mathbb{R}^2}\|_{L^1(\mathbb{R}^2)} = \|\vec{G}^n(U^0) - \vec{G}^m U^0)\|_{L^1(\Delta)}$$

$$= \|\sum_{i=0}^{n-m-1} (\vec{G}^{n-i}(U^0) - \vec{G}^{n-(i+1)}(U^0))\|_{L^1(\Delta)}$$

$$\le \sum_{i=0}^{n-m-1} \|\vec{G}^{n-i}(U^0) - \vec{G}^{n-(i+1)}(U^0)\|_{L^1(\Delta)}$$

$$\le \sum_{i=0}^{n-m-1} \|\vec{G}(U^0) - U^0\|_{L^1(\Delta)} = (n-m)\|\vec{G}(U^0) - U^0\|_{L^1(\Delta)} ,$$

which gives the desired result.

We next need to estimate the quantity $\|\vec{G}(U^0) - U^0\|_{L^1(\Delta)}$.

<u>Proposition 3.5</u>. Let G be given by (0.4) where the fluxes g_1, g_2 are Lipschitz continuous. Then

$$\frac{1}{\Delta t}\|\vec{G}(U) - U\|_{L^1(\Delta)} \le L(p+q+2)(r+s+2)\|U\|_{BV(\Delta)}$$

where L is a Lipschitz constant for g_1, g_2.

<u>Proof</u>. By (0.4)

$$\|\vec{G}(U) - U\|_{L^1(\Delta)} \le \sum_{j,k} (\lambda^x|g_1(U_{j-p+1,k-r},\cdots,U_{j+q+2,k+s+1})$$

$$- g_1(U_{j-p,k-r},\cdots,U_{j+q+1,k+s+1})| + \lambda^y|g_2(U_{j-p,k-s+1},\cdots,U_{j+q+1,k+s+1}) -$$

$$g_2(U_{j-p,k-s},\cdots,U_{j+q+1,k+s+2})|)\Delta x\Delta y$$

$$\le \sum_{j,k}\left(\frac{\Delta t}{\Delta x} L \sum_{\ell=j-p}^{j+q+1}\sum_{m=k-r}^{k+s+1} |U_{\ell+1,m} - U_{\ell,m}| + \frac{\Delta t}{\Delta y} L \sum_{\ell=j-p}^{j+q+1}\sum_{m=k-r}^{k+s+1} |U_{\ell,m+1} - U_{\ell,m}|\right)\Delta x\Delta y$$

$$\le (\Delta t)L(p+q+2)(r+s+2)\left(\frac{1}{\Delta x}\|\Delta^x_+ U\|_{L^1(\Delta)} + \frac{1}{\Delta y}\|\Delta^y_+ U\|_{L^1(\Delta)}\right)$$

$$= (\Delta t)L(p+q+2)(r+s+2)\|U\|_{BV(\Delta)} ,$$

whence the result.

The following corollary of these estimates will be what is eventually used.

<u>Corollary 3.6</u>. Let the assumptions of Propositions 3.4 and 3.5 hold. Let $u_0 \in L^1(\mathbb{R}^2)$, $a \le u_0 \le b$ a.e., and $u^{\Delta t}$ be given by (3.3) where $U^0 = u_{0\Delta}$ (see (2.13)). Then there is a constant C independent of u_0, Δt such that for $0 \le t_1, t_2$

$$\int_{\mathbb{R}^2} |u^{\Delta t}(x,y,t_2) - u^{\Delta t}(x,y,t_1)|\,dxdy \le C(|t_1 - t_2| + \Delta t)\|u_0\|_{BV(\mathbb{R}^2)} .$$

<u>Proof</u>. This is immediate from Propositions 3.4, 3.5 and (2.17).

<u>Proof of Lemma 3.2</u>. First we prove that (a) implies (b). Let $f, g \in C$. Then $f \vee g = g + (f-g)^+ \in C$ by assumption and $T(f \vee g) - T(g) \ge 0$ since T is order preserving. Also $T(f) - T(g) \le T(f \vee g) - T(g)$ and so $(T(f) - T(g))^+ \le T(f \vee g) - T(g)$. Thus

$$\int_\Omega (T(f) - T(g))^+ \le \int_\Omega (T(f\vee g) - T(g)) = \int_\Omega (f\vee g - g) = \int_\Omega (f-g)^+$$

and (b) is established. That (b) implies (c) is trivial. Indeed, if (b) hols,

$$\int_\Omega |T(f) - T(g)| = \int_\Omega (T(f) - T(g))^+ + \int_\Omega (T(g) - T(f))^+ \le \int_\Omega (f-g)^+ + \int_\Omega (g-g)^+$$
$$= \int_\Omega |f-g| \ .$$

Finally, if $f, g \in C$, $f \ge g$ and (c) holds $2s^+ = |s| + s$ implies

$$2\int_\Omega (T(g) - T(f))^+ = \int_\Omega |T(g) - T(f)| - \int_\Omega (T(g) - T(f)) \le \int_\Omega |g-f| - \int_\Omega (g-f)$$
$$= 0 \ .$$

We have proved Lemma 3.2 here for completeness. The parallel result for L^∞ and extensions are discussed in [5]. It is recognized that there will be results analogous to those of this paper for, in particular, equations of the form $u_t - \Delta\varphi(u) = 0$ and $u_t + f(\text{grad } u) = 0$ and, time permitting, these will be developed.

<u>Section 4</u>. The Entropy Contion

In this section we establish that if $u^{\Delta t}$ is an approximation of a solution of (0.1) produced by a monotone conservation-form difference scheme via the prescription (3.3) and there is a sequence $\Delta t_\ell \to 0$ for which $u^{\Delta t_\ell}$ converges to a limit u boundedly a.e., then u is an entropy solution of (0.1). In the next section we show that every sequence Δt_ℓ convergent to zero has a subsequence with the above property and then deduce that $u^{\Delta t}$ converges as $\Delta t \to 0$ from the uniqueness Theorem 2.1.

If Δt_ℓ is given, it determines the lattices $\{(j\Delta x, k\Delta y)\} = \{(j/\lambda^x, k/\lambda^y)\Delta t_\ell\}$, $\{(j\Delta x, k\Delta y, n\Delta t_\ell)\} = \{(j/\lambda_x, k/\lambda^y, n)\Delta t_\ell\}$ and the associated partitions $\{R_{j,k}\}$, $\{R_{j,k} \times [n\Delta t_\ell, (n+1)\Delta t_\ell)\}$ of $\mathbb{R}^2$ and $\mathbb{R}^2 \times [0,\infty)$. These <u>depend on</u> <u>ℓ</u>, but this will not be explicitly indicated by our notation. The ratios λ^x, λ^y will be held constant (so $\Delta x, \Delta y$ depend on ℓ). As ℓ varies so too will the initial data involved in computing $u^{\Delta t_\ell}$, but this will not be indicated either.

<u>Proposition 4.1</u>. Let (3.1) be a conservation form difference approximation consistent with (0.1) which is monotone on $[a,b]$ and has continuous numerical fluxes g_1, g_2. Suppose a sequence Δt_ℓ of positive numbers convergent to 0 is given. Let $u^{\Delta t_\ell}$ be computed by (3.3), $a \le U^0_{j,k} \le b$, and $u^{\Delta t_\ell} \to u$ boundedly a.e. on $\mathbb{R}^2 \times [0,T]$. Then u is an entropy solution of $u_t + f_1(u)_x + f_2(u)_y = 0$ on $\mathbb{R}^2 \times [0,T]$.

Lax and Wendroff [15] observed that under the above assumptions u will be a weak solution of the equation (even if G is not monotone). We require the following simple lemma which is at the crux of their argument:

<u>Lemma 4.2</u>. Let $\Delta t_\ell \to 0$ and

$$\lim_{\ell\to\infty} \sum_{n=0}^{\infty} \sum_{j,k} V^n_{j,k} \chi^n_{j,k} = V$$

boundedly a.e. on $\mathbb{R}^2 \times [0,\infty)$. Let

$$\varphi \in C^1_0(\mathbb{R}^2 \times (0,\infty)) \quad \text{and} \quad \varphi^n_{j,k} = \varphi(j\Delta x, k\Delta y, n\Delta t_\ell) \ .$$

Then

$$\lim_{\ell\to\infty} \sum_{n=0}^{\infty} \sum_{j,k} \phi^n_{j,k} (\Delta^z_+ v)^n_{j,k} \Delta x \Delta y \Delta t_\ell = - \int_0^\infty \int_{\mathbb{R}^2} \left(\frac{\partial \phi}{\partial z}\right) v \, dx dy dt$$

for $z = x,y,t$.

The proof is elementary and is omitted. (While this lemma is not formulated in [15], the arguments there suffice for the proof.)

Proof of Proposition 4.1. Our proof follows the strategy of the one given by Harten, Hyman and Lax [10] for $N = 1$. Given $c \in \mathbb{R}$ we will produce continuous numerical entropy fluxes $h_1(a_{-p,-r},\ldots,a_{q,s+1})$, $h_2(a_{-p,-r},\ldots,a_{q+1,s})$ such that $h_i(u,\cdots,u) = \mathrm{sgn}(u-c)(f_i(u) - f_i(c))$ and

$$(4.2) \qquad \Delta^t_+|U^n_{j,k} - c| + \lambda^x \Delta^x_+ h_1(U^n_{j-p,k-r},\ldots,U^n_{j+q,k+s+1}) + \lambda^y \Delta^y_+ h_2(U^n_{j-p,k-r},\ldots,U^n_{j+q+1,k+s}) \le 0 .$$

Once this is done, Proposition 4.1 follows. Indeed, multiply (4.2) by $\Delta x \Delta y \Delta t_\ell \, \phi^n_{j,k} \ge 0$ and sum. Setting, respectively, $v^n_{j,k} = |U^n_{j,k} - c|$, $h_1(U^n_{j-p,k-r},\ldots,U^n_{j+q,k+s+1})$, $h_2(U^n_{j-p,k-r},\ldots,U^n_{j+q+1,k+s})$ we can take, respectively, $v = |u - c|$, $\mathrm{sgn}(u - c)(f_1(u) - c)$, $\mathrm{sgn}(u - c)(f_2(u) - c)$ in Lemma 4.1. Letting $\ell \to \infty$ and using Lemma 4.1 yields

$$- \int_0^\infty \int_{\mathbb{R}^2} \phi_t|u-c| + \phi_x \mathrm{sgn}(u-c)(f_1(u)-f_1(c)) + \phi_y \mathrm{sgn}(u-c)(f_2(u)-f_2(c))\, dx dy dt \le 0$$

which is the entropy condition (2.2).

It remains to produce h_1, h_2. Here we improve on [10] in simplicity and generality. With $z \vee w = \max(z,w)$, $z \wedge w = \min(z,w)$ we set

$$(4.3) \qquad \begin{cases} h_1(U_{j-p,k-r},\ldots,U_{j+q,k+s+1}) = g_1(c\vee U_{j-p,k-r},\ldots,c\vee U_{j+q,k+s+1}) \\ \quad - g_1(c\wedge U_{j-p,k-r},\ldots,c\wedge U_{j+q,k+s+1}) \\ h_2(U_{j-p,k-r},\ldots,U_{j+q+1,k+s}) = g_2(c\vee U_{j-p,k-r},\ldots,c\vee U_{j+q+1,k+s}) \\ \quad - g_2(c\wedge U_{j-p,k-r},\ldots,c\wedge U_{j+q+1,k+s}) . \end{cases}$$

A direct calculation using only the definitions yields the identity:

$$(4.4) \qquad |U^n_{j,k} - c| - \lambda^x \Delta^x_+ h_1(U^n_{j-p,k-r},\ldots) - \lambda^y \Delta^y_+ h_2(U^n_{j-p,k-r},\ldots)$$
$$= G(c\vee U^n_{j-p,k-r},\ldots,c\vee U^n_{j+q+1,k+s+1})$$
$$- G(c\wedge U^n_{j-p,k-r},\ldots,c\wedge U^n_{j+q+1,k+s+1}) .$$

We also have the relations

$$(4.5) \qquad \begin{cases} U^{n+1}_{j,k} = G(U^n_{j-p,k-r},\ldots,U^n_{j+q+1,k+s+1}) \\ c = G(c,c,\cdots,c) . \end{cases}$$

From (4.5) and the monotonicity of G we find

$$(4.6) \qquad \begin{cases} c\vee U^{n+1}_{j,k} \le G(c\vee U^n_{j-p,k-r},\ldots,c\vee U^n_{j+q+1,k+s+1}) \\ -(c\wedge U^{n+1}_{j,k}) \le - G(c\wedge U^n_{j-p,k-r},\ldots,c\wedge U^n_{j+q+1,k+s+1}) . \end{cases}$$

Adding the inequalities (4.6) and entering the result in (4.4) yields (4.2). This completes the proof.

Section 5. The Proof of Theorem 1.

We will actually prove more than Theorem 1 states here, for we will not need to assume the existence of the solution $S(t)u_0$ of (0.1). This will follow from our proofs. To begin, we assume that $a \le u_0 \le b$ a.e., $u_0 \in BV(\mathbb{R}^2)$ and u_0 has compact support. Let $u^{\Delta t}$ be given by (3.3) with $U^0 = u_{0\Delta}$ (see (2.13)).

From the stability estimates of Proposition 3.1 and (2.14) - (2.17) we deduce

$$(5.1) \qquad \begin{cases} \|u^{\Delta t}(\cdot,\cdot)t)\|_Z \le \|u_0\|_Z \quad \text{for} \quad Z = L^1(\mathbb{R}^2),\ (BV(\mathbb{R}^2), \text{ and} \\ \quad a \le u^{\Delta t} \le b \quad \text{everywhere.} \end{cases}$$

Also, because λ^x, λ^y are fixed, it is easy to see that there is a $c_0 > 0$ such that if u_0 vanishes for $|x| + |y| \ge R$, then

$$(5.2) \qquad u^{\Delta t}(x,y,t) = 0 \quad \text{for} \quad |x| + |y| \ge R + c_0 + c_0 t, \quad 0 \le \Delta t \le 1 \ .$$

Because bounded subsets of $BV(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ are precompact in $L^1_{loc}(\mathbb{R}^2)$ (see (2.9)), (5.1) and (5.2) imply that if $T > 0$ then

$$(5.3) \qquad \{u^{\Delta t}(\cdot,\cdot,t) : 0 \le t \le T,\ 0 \le \Delta t \le 1\} \quad \text{is precompact in} \quad L^1(\mathbb{R}^2) \ .$$

Next the estimate of Corollary 3.5 supplies us with

$$(5.4) \qquad \|u^{\Delta t}(\cdot,\cdot,t_1) - u^{\Delta t}(\cdot,\cdot,t_2)\|_{L^1(\mathbb{R}^2)} \le c(|t_1-t_2| + 2\Delta t)\|u_0\|_{BV(\mathbb{R}^2)} \ .$$

By the proof of Arzela-Ascoli's theorem, (5.3) and (5.4) guarantee that if $\Delta t_\ell \to 0$ then there is a subsequence $\Delta t_{\ell(k)}$ and a function $u : [0,\infty) \to L^1(\mathbb{R}^2)$ such that

$$(5.5) \qquad \lim_{k\to\infty} \max_{0\le t\le T} \|u^{\Delta t_{\ell(k)}}(\cdot,\cdot,t) - u(\cdot,\cdot,t)\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{for} \quad T > 0.$$

From (5.1) and (5.4) we deduce $a \le u \le b$ a.e. and

$$(5.6) \qquad \|u(\cdot,\cdot,t_1) - u(\cdot,\cdot,t_2)\|_{L^1(\mathbb{R}^2)} \le c|t_1-t_2|\,\|u_0\|_{BV(\mathbb{R}^2)} \ .$$

Passing to a further subsequence if necessary, we can have $u^{\Delta t_{\ell(k)}} \to u$ boundedly a.e. By Proposition 4.1, u is an entropy solution of the conservation law in (0.1). By (5.6), and

$u(\cdot,\cdot,0) = u_0$, u assumes the initial value u_0 continuously in $L^1(\mathbb{R}^2)$. The uniqueness Theorem 2.1 guarantees then the uniqueness of u, and we conclude

$$(5.7) \qquad \lim_{\Delta t\to 0} \|u^{\Delta t}(\cdot,\cdot,t) - u(\cdot,\cdot,t)\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{uniformly for bounded} \quad t \ .$$

This completes the analysis when u_0 has compact support.

For general $u_0 \in L^1(\mathbb{R}^2)$ with $a \le u_0 \le b$ a.e. we choose $u_{0,m} \in BV(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ with compact support such that

$$(5.8) \qquad \begin{cases} a \le u_{0,m} \le b \quad \text{and} \\ \|u_0 - u_{0,m}\|_{L^1(\mathbb{R}^2)} \to 0 \quad \text{as} \quad m \to \infty \ . \end{cases}$$

This can be done by any standard method. Denote the difference scheme solution corresponding to the initial-value $u_{0,m}$ by $u_m^{\Delta t}$. Below we regard all functions as function of t with values in $L^1(\mathbb{R}^2)$.) We know that

$$(5.9) \qquad \lim_{\Delta t\to 0} u_m^{\Delta t}(t) = u_m(t) \quad \text{exists uniformly for bounded} \quad t \ .$$

Moreover, from Proposition 3.1(c) it follows that

$$\sup_{t\ge 0} \|u_m^{\Delta t}(t) - u_\ell^{\Delta t}(t)\|_{L^1(\mathbb{R}^2)} \le \|u_{0,m} - u_{0,\ell}\|_{L^1(\mathbb{R}^2)}$$

and so $\{u_m\}$ is Cauchy in $C([0,\infty) : L^1(\mathbb{R}^2))$ and hence converges uniformly to a limit $u \in C([0,\infty) : L^1(\mathbb{R}^2))$. Finally,

$$\begin{aligned} \|u^{\Delta t}(t) - u(t)\|_{L^1(\mathbb{R}^2)} &\le \|u^{\Delta t}(t) - u_m^{\Delta t}(t)\|_{L^1(\mathbb{R}^2)} \\ &+ \|u_m^{\Delta t}(t) - u_m(t)\|_{L^1(\mathbb{R}^2)} + \|u_m(t) - u(t)\|_{L^1(\mathbb{R}^2)} \\ &\le \|u_0 - u_{0,m}\|_{L^1(\mathbb{R}^2)} + \|u_m^{\Delta t}(t) - u_m(t)\|_{L^1(\mathbb{R}^2)} \\ &+ \|u_m(t) - u(t)\|_{L^1(\mathbb{R}^2)} \ . \end{aligned}$$

The first and third terms above can be made small uniformly in t by taking m large. Then the middle term can be made small by taking Δt small (uniformly for t bounded). Hence

$\lim_{\Delta t \to 0} u^{\Delta t} = u$ locally uniformly in $L^1(\mathbb{R}^2)$ (and u is continuous into $L^1(\mathbb{R}^2)$ by its construction). This completes the proof.

A reexamination of the proof of Theorem 5.1 and the various ingredients of the preceding sections shows that we have in fact proved the following results on existence and properties of solutions of (0.1) (given the uniqueness Theorem 2.1).

Corollary 5.1. Let the conservation law have Lipschitz continuous fluxes f_i. Then for every initial-data $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ there is a unique entropy solution $u \in C([0,\infty) : L^1(\mathbb{R}^N))$ of (0.1) with $u(0) = u_0$. Denoting this solution by $S(t)u_0$ we have:

(a) If $u_0 \in BV(\mathbb{R}^N)$, $t \to S(t)u_0$ is Lipschitz continuous into $L^1(\mathbb{R}^N)$ and

$$\|S(t)u_0\|_{BV(\mathbb{R}^N)} \le \|u_0\|_{BV(\mathbb{R}^N)} .$$

(b) $\|S(t)u_0 - S(t)v_0\|_{L^1(\mathbb{R}^N)} \le \|u_0 - v_0\|_{L^1(\mathbb{R}^N)}$.

(c) $u_0 \le v_0$ a.e. implies $S(t)u_0 \le S(t)v_0$ a.e.

(d) $a \le u_0 \le b$ a.e. implies $a \le S(t)u_0 \le b$ a.e.

All of the above is well-known, even in much greater generality. (See [13] in particular). Crandall [4] and Benilan [1] also treat cases in which the f_i need not be Lipschitz. In fact, it is enough that the f_i are continuous and

$$\lim_{r \to 0} |f_i(r) - f_i(0)|/|r|^{(N-1)/N} = 0$$

in order that Theorem 2.1 hold with $u_{10} \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ provided $R = \infty$ in (2.3) and (2.4). In order to compute solutions of (0.1) in this non-Lipschitz case, one would have to approximate $(f_1,\cdots,f_N)$ by smoother functions $(f_1^\ell,\cdots,f_N^\ell)$, solve a difference approximation to the resulting problem with $\lambda_\ell^x, \lambda_\ell^y$ choosen appropriately and then let $\ell \to \infty$, $\Delta t \to 0$. The modulus of continuity in time must be treated appropriately, but we will not consider this here.

Section 6. The Inhomogeneous Equation

We briefly remark on how the analysis given above can be carried out for the more general problem

$$(6.1) \qquad \begin{cases} u_t + \sum_{i=1}^{N} f_i(u)_{x_i} = F(x,t) \\ u(x,0) = u_0(x) . \end{cases}$$

The corresponding difference schemes have the form (for $N = 2$)

$$(6.2) \qquad U_{j,k}^{n+1} = G(U_{j-p,k-r}^n,\cdots,U_{j+q+1,k+s+1}^n) + \Delta t F_{j,k}^n ,$$

or

$$(6.3) \qquad U^{n+1} = \bar{G}(U^n) + \Delta t F^n ,$$

where G satisfies the conditions of the preceding sections.

Theorem 6.1. Let G define a conservation form difference scheme consistent with (0.1), which is monotone on $[a,b]$ and has Lipschitz continuous fluxes. Let $T > 0$, $u_0 \in L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ and $F \in L^\infty(\mathbb{R}^2 \times (0,T)) \cap L^1(\mathbb{R}^2 \times (0,T))$ satisfy $a + T\|F\|_{L^\infty} \le u_0 \le b - \|F\|_{L^\infty} T$ a.e. Let $U^0 = u_{0\Delta}$ and

$$F_{j,k}^n = \frac{1}{\Delta t \Delta x \Delta y} \int_{n\Delta t}^{(n+1)\Delta t} \int_{R_{j,k}} F(x,y,t)\,dx\,dy\,dt .$$

Then U^n, defined by (6.3) satisfies $a \le U_{j,k}^n \le b$ for all n,j,k such that $n\Delta t \le T$ and

$$u^{\Delta t} = \sum_{n=0}^{\infty} \sum_{j,k} U_{j,k}^n \chi_{j,k}^n$$

converges in $L^1(\mathbb{R}^2)$ uniformly on compact subsets of $[0,T)$ to the unique entropy solution of (6.1).

Sketch of Proof: In addition to the arguments in preceding sections we require a uniqueness theorem (see [13] for this) and a few estimates given now. From Proposition 3.1 we deduce immediately

$$(6.4)\qquad \begin{cases} \sup_{j,k} U^{n+1}_{j,k} \le \sup_{j,k} U^n_{j,k} + \Delta t \sup_{j,k} F^n_{j,k} \\ \inf_{j,k} U^{n+1}_{j,k} \ge \inf_{j,k} U^n_{j,k} + \Delta t \inf_{j,k} F^n_{j,k} \end{cases}$$

and

$$(6.5)\qquad \|U^{n+1}\|_X \le \|U^n\|_X + \Delta t \|F^n\|_X$$

for $X = L^1(\Delta)$ and $X = BV(\Delta)$. Moreover, if $\hat{U}^{n+1} = G(\hat{U}^n) + \Delta t\, \hat{F}^n$ we find

$$(6.6)\qquad \|U^n - \hat{U}^n\|_{L^1(\Delta)} \le \|U^0 - \hat{U}^0\|_{L^1(\Delta)} + \sum_{j=0}^{n-1} \Delta t \|F^j - \hat{F}^j\|_{L^1(\Delta)} .$$

Finally, the estimate of Proposition 3.4 is replaced (with $n\Delta t \le t_2 < (n+1)\Delta t$, $m\Delta t \le t_1 < (m+1)\Delta t$ and $m \le n$ as before) by

$$\int_{\mathbb{R}^2} |u^{\Delta t}(x,y,t_2) - u^{\Delta t}(x,y,t_1)|\,dt$$

$$= \|U^n - U^m\|_{L^1(\Delta)} \le \sum_{j=m}^{n-1} \|U^{j+1} - U^j\|_{L^1(\Delta)}$$

$$= \sum_{j=m}^{n-1} \|\vec{G}(U^j) + \Delta t F^j - U^j\|_{L^1(\Delta)}$$

$$\le \sum_{j=m}^{n-1} \|\vec{G}(U^j) - U^j\|_{L^1(\Delta)} + \sum_{j=m}^{n-1} \Delta t \|F_j\|_{L^1(\Delta)} .$$

Now we use Proposition 3.5 and (6.5) to estimate the right-hand side above by

$$\Delta t(n-m)\ \text{cons.}(\|U^0\|_{BV(\Delta)}) + \sum_{j=0}^{n-1} \Delta t \|F^j\|_{BV(\Delta)}) + \sum_{j=m}^{n-1} \Delta t \|F_j\|_{L^1(\Delta)} .$$

Thus if

$$\int_0^T \|F(\cdot,\cdot,t)\|_{BV(\mathbb{R}^2)}\, dt < \infty$$

we have the type of equicontinuity of $u^{\Delta t}$ as $\Delta t \to 0$ used in the proof of Theorem 1, and the proof may be completed much as before. (One begins, say, with $u_0 \in C_0^\infty(\mathbb{R}^2)$, $F \in C_0^\infty(\mathbb{R}^2 \times [0,T))$ and passes to the general case via (6.6).)

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A complete self-contained treatment of the stability and convergence properties of conservation form, monotone difference approximations to scalar conservation laws in several space variables is developed. In particular, the authors prove that general monotone difference schemes always converge and that they converge to the physical weak solution satisfying the entropy condition. Rigorous convergence results follow for dimensional splitting algorithms when each step is approximated by a monotone difference scheme. → next page

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ABSTRACT (continued)

The results are general enough to include, for instance, Godunov's scheme, the upwind scheme (differenced through stagnation points), and the Lax-Friedrichs scheme together with appropriate multi-dimensional generalizations.